Stability and Bifurcation of Homogeneous Deformations of a Compressible Elastic Body Under Pressure Load

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Abstract: An analysis is given of the stability and bifurcation of homogeneous deformations of a compressible homogeneous isotropic elastic body under pressure loads. The energy minimum principle is used in the stability analysis and singularity theory in the bifurcation analysis. It is found that multiple equilibrium deformations are possible under pressure loads that describe the Cauchy traction over the boundary of the deformed body.

I. INTRODUCTION

The problem of stability and bifurcation of homogeneous deformations of elastic materials has attracted the attention of many researchers as it provides a stage for the development of mathematical theories of nonlinear elasticity. A well-known example is the study of the stability of homogeneous equilibrium deformations of an elastic cube by Rivlin [10, 11]. The cube was assumed to be made of incompressible neo-Hookean material and subjected to three pairs of equal and opposite dead-load surface tractions. Rivlin found that as the magnitude of the tensile tractions exceeds a certain value, the identity deformation loses stability, and there exist asymmetric homogeneous equilibrium deformations, some of which are stable in the sense that they minimize the total energy locally. These asymmetric deformations have two equal principal stretches that are larger than unity. Hence, the cube is deformed into a plate under dead-load tractions of equal magnitude over its six faces.

This prominent feature has stimulated the interest of other researchers. The problem has been studied further by Pipkin [9], Sawyers [12], Ball and Shaeffer [1], Ogden [7], and others, with more general constitutive relations and more powerful mathematical tools.

In this problem, and in other related problems, the dead-load traction is an important assumption. As pointed out by many authors, this assumption renders great mathematical convenience in formulating and solving traction boundary value problems. Of course, dead loads are not the only type of boundary tractions possible. They are not even the boundary tractions readily available experimentally. In fact, it is very difficult to produce perfect dead-load tractions in laboratories. Rivlin’s [10] solutions have been known for nearly a half century and have been reproduced in numerous publications in nonlinear elasticity. Nevertheless, we have yet to find an experiment that demonstrates these solutions.

In a recent paper, Hoger [6] used the Rivlin’s [11] cube problem in a discussion of dead-load boundary value problems. She believed that the dead-load traction boundary condition “does not correspond to an unambiguous physical situation: more than one pair of deformation and actual applied traction on the deformed body can produce the same Piola-Kirchhoff stress vector on the undeformed body.” [6, p. 1] She argued further that “the occurrence of these
multiple solutions does not appear unusual when the solutions are interpreted in terms of the physically meaningful Cauchy tractions on the deformed body." Among other things, Hoger’s argument calls for study of multiple equilibrium solutions under other traction boundary conditions, preferably in terms of Cauchy tractions.

Pressure is one of the most common boundary tractions in practice. It assigns the Cauchy traction as a vector normal to the boundary of the deformed body. Although there is a vast literature on the finite deformations of elastic membranes under pressure loads, the study of bifurcation and stability of the deformations of three-dimensional elastic bodies under such loads is scarce. Here we cite the work of Ogden [7, Section 6.2.6], who used the techniques of incremental equations to derive necessary conditions for bifurcations.

The purpose of the present work is to study the stability and bifurcation of deformations of a general isotropic compressible elastic body under uniform pressure loads. It is found that in spite of the difference in the nature of boundary tractions, the mathematical structure of the bifurcation equations and the solutions of the present problem have considerable similarity to those of the dead-load problem. In particular, there can exist multiple equilibrium solutions that correspond to the same Cauchy stress vector on the deformed body.

In Section II, we formulate the stability problem as the minimization of an energy functional. The first variation condition of this minimization problem leads to the usual equilibrium equation and pressure boundary condition. The existence and characterization of stable deformations are discussed in Section III. It is shown that under certain growth conditions on the strain-energy function, there always exists a stable deformation that is homogeneous. This enables us to focus on this particular class of deformations. Global and local stability conditions are derived. Section IV is devoted to the bifurcation analysis of homogeneous deformations. The main tool for the analysis is singularity theory, which is used to derive both necessary and sufficient conditions for bifurcation. Symmetry properties of the problem are systematically utilized to facilitate the analysis. Stability of the solution branches is studied by using the results obtained in Section III.

II. BASIC FORMULAE

We consider a body of a compressible elastic material, which occupies a regular bounded domain $\Omega \subset \mathbb{R}^3$ in a reference configuration. The results of this work are independent of the geometry of $\Omega$. However, it helps to visualize the problem if $\Omega$ is assumed to be a unit cube. A representative material particle of the body is denoted by $X \in \Omega$.

In this work, we consider the class of $C^1$ deformations. The set of kinematically admissible deformations is then given by

$$\mathcal{A} \equiv \{ x \in C^1(\Omega; \mathbb{R}^3) : \det \nabla x > 0 \}.$$

We denote the deformation gradient $\nabla x$ by $F$. The principal stretches $\lambda_i$, $i = 1, 2, 3$, are the eigenvalues of the stretch tensor $\sqrt{F^T F}$.

The material is assumed to be homogeneous and isotropic in the reference configuration. The constitutive behavior of the material is described by a strain-energy function $W(F)$, which is assumed to be of class $C^4$. Because of isotropy, the strain energy depends on the deformation gradient only through the principal stretches:

$$W(F) = \hat{W}(\lambda_1, \lambda_2, \lambda_3).$$
The reduced strain-energy function \( \hat{W} \) is symmetric in its arguments. This symmetry can be described by the invariance under the permutation group \( S_3 \), as will be discussed below.

The body is subjected to a uniform pressure \( p \) over its boundary, whose value is controlled by a loading device. In spite of the term \textit{pressure}, the value of \( p \) is not necessarily positive, although such is the case in most physical situations. The body and the loading device form a conservative system associated with the total energy functional

\[
E[x] = \int_{\Omega} \left[ W(\nabla x) + p \, \det \nabla x \right] dV. \tag{1}
\]

The second term on the right-hand side of (1) corresponds to the potential of the pressure load, which is assumed to be the product of \( p \) and the deformed volume of the body.

By the energy stability criterion, a deformation \( x \in \mathcal{A} \) is globally stable if it minimizes the total energy in \( \mathcal{A} \), that is, if

\[
E[x] \leq E[x^*] \quad \forall \ x^* \in \mathcal{A}. \tag{2}
\]

A deformation \( x \) is locally stable if (2) holds for all \( x^* \) in a neighborhood of \( x \). The notion of local stability obviously depends on the topology used in defining the neighborhood.

In the literature, the deformation \( x \) is sometimes said to be neutrally stable if the equality in (2) holds for some \( x^* \neq x \), whereas the term \textit{stable deformations} is reserved for those \( x \) that satisfy the strict inequality of (2) for all \( x^* \neq x \). If these terminologies are used, a deformation in the present problem is at most neutrally stable because any rigid deformation leaves \( E \) invariant. The usage of these terminologies appears to be more physically meaningful if deformations that differ by rigid transformations are regarded as equivalent. We shall comment on this further in the next section.

In the literature on dynamical systems, stability is usually defined by dynamical criteria, which are not equivalent to, but related to the static criterion used in the present work. In particular, it has been shown [5] that the asymptotic stability of an equilibrium state of a dynamical system is implied by the so-called linear stability, which requires that the eigenvalues of the associated linear operator all have positive real parts. It turns out that this latter condition is also sufficient for an equilibrium deformation to be stable under the present energy criterion. Furthermore, in the dynamical systems literature, the neutral stability induced by a symmetry, like those here induced by rigid transformations, is treated with the notion of linear orbital stability, which requires that those eigenvalues that are not forced by the symmetry to be zero have positive real parts.

The first variation condition of the minimization of functional (1) yields the familiar equilibrium equation of zero body force and the pressure boundary condition:

\[
\text{Div} \frac{\partial W}{\partial \mathbf{F}} = 0 \quad \text{in} \ \Omega,
\]

\[
\frac{\partial W}{\partial \mathbf{F}} \mathbf{N} = -p(\det \mathbf{F})\mathbf{F}^{-T} \mathbf{N} \quad \text{on} \ \partial \Omega. \tag{3}
\]

Obviously, the boundary tractions are not dead-load since the right-hand side of (3) depends on the deformation gradient \( \mathbf{F} \).
III. EXISTENCE AND CHARACTERIZATION OF STABLE DEFORMATIONS

A deformation is homogeneous if the deformation gradient is constant in \( \Omega \), and is pure homogeneous if, in addition, the deformation gradient is symmetric and hence can be made to be of diagonal form in a rectangular coordinate system. We first show that under certain conditions, there exists, for any given load, a globally stable deformation that is purely homogeneous.

**Proposition 1.** Suppose that

\[
\lim_{(\lambda_1,\lambda_2,\lambda_3) \to (1,1,1)} \hat{W}(\lambda_1, \lambda_2, \lambda_3) = \infty, \quad \lim_{\left|\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}\right)\right| \to \infty} \hat{W}(\lambda_1, \lambda_2, \lambda_3) = \infty.
\]

(4)

Then, for any \( p \geq 0 \), the total energy \( E \) attains its minimum at a pure homogeneous deformation

\[
x_i = \lambda_i X_i, \quad i \text{ no sum}.
\]

(5)

Moreover, if

\[
\lim_{(\lambda_1,\lambda_2,\lambda_3) \to (1,1,1)} \frac{\hat{W}(\lambda_1, \lambda_2, \lambda_3)}{\lambda_1 \lambda_2 \lambda_3} = \infty
\]

and (4) hold, then for any \( p < 0 \), the total energy attains its minimum at a pure homogeneous deformation.

**Proof.** Define a smooth function \( \phi: \mathbb{R}_+^3 \times \mathbb{R} \to \mathbb{R} \) by

\[
\phi(\lambda_1, \lambda_2, \lambda_3, p) \equiv \hat{W}(\lambda_1, \lambda_2, \lambda_3) + p \lambda_1 \lambda_2 \lambda_3.
\]

(6)

Then

\[
W(\nabla x) + p \det \nabla x = \phi(\lambda_1, \lambda_2, \lambda_3, p),
\]

with \( \lambda_i \) being the principal stretches of \( x \). Now it suffices to show that \( \phi(\cdot, p) \) attains its minimum at some \( (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}_+^3 \). Let a \( p \geq 0 \) be given. Conditions (4) ensure that \( \phi(\cdot, p) \) is bounded below on \( \mathbb{R}_+^3 \), and that a minimizing sequence of \( \phi \) is contained in a compact subset of \( \mathbb{R}_+^3 \). The desired result then follows. The proof for the case where \( p < 0 \) is similar. \( \square \)

Proposition 1 suggests that when looking for stable deformations we concentrate on pure homogeneous deformations. The next proposition shows that when examining the stability of a pure homogeneous deformation, the minimization of the total energy functional \( E \) reduces to that of the scalar energy function \( \phi \).

**Proposition 2.** The pure homogeneous deformation (5) is globally stable if and only if

\[
\phi(\lambda_1, \lambda_2, \lambda_3, p) \leq \phi(\lambda_1^*, \lambda_2^*, \lambda_3^*, p) \quad \forall \, (\lambda_1^*, \lambda_2^*, \lambda_3^*) \in \mathbb{R}_+^3.
\]

(7)

**Proof.** The conclusion follows from the definition of global stability and the observation that

\[
E[x] = \int_\Omega \phi(\lambda_1, \lambda_2, \lambda_3, p) \, dV.
\]

(8)

The above proposition reveals an important feature of the present problem. At a homogeneous deformation, the minimization problem in an infinite-dimensional function space reduces to one in a three-dimensional vector space. This is a consequence of the fact that the total energy functional depends on the deformation only through the principal stretches. For
some other types of loads, for example dead-load tractions, the total energy is dependent on the full nine-dimensional tensor space of deformation gradients, and the use of derivative tests for local stability involves computing the derivatives of the strain-energy function \( W(F) \), in contrast to computing the derivatives of the reduced strain-energy function \( \hat{W}(\lambda_1, \lambda_2, \lambda_3) \) in the present case.

As another remark, we note that the principal stretches are invariant under rigid deformations. Thus, the neutral stability induced by rigid deformations is effectively eliminated if one requires a neutrally stable pure homogeneous deformation (5) to satisfy (7) with the equality holding for some \( (\lambda_1^*, \lambda_2^*, \lambda_3^*) \) unobtainable from \( (\lambda_1, \lambda_2, \lambda_3) \) by permutations, and a stable pure homogeneous deformation to satisfy the strict inequality (7) for all \( (\lambda_1^*, \lambda_2^*, \lambda_3^*) \) unobtainable from \( (\lambda_1, \lambda_2, \lambda_3) \). These definitions are related to the notion of orbital stability in the dynamical systems literature.

In the next two propositions, we give further characterization of stable pure homogeneous deformations.

**Proposition 3.** The pure homogeneous deformation (5) is globally stable under pressure \( p \) if and only if

\[
\lambda_i \hat{W}_i(\lambda_1, \lambda_2, \lambda_3) + p\lambda_1\lambda_2\lambda_3 = 0 \quad i \text{ no sum},
\]

(8)

\[
\hat{W}(\lambda_1^*, \lambda_2^*, \lambda_3^*) \geq \hat{W}(\lambda_1, \lambda_2, \lambda_3)
\]

\[
+ \left( \frac{\lambda_1^*\lambda_2^*\lambda_3^*}{\lambda_1\lambda_2\lambda_3} - 1 \right) \lambda_i \hat{W}_i(\lambda_1, \lambda_2, \lambda_3) \forall (\lambda_1^*, \lambda_2^*, \lambda_3^*) \in \mathbb{R}_+^3, \quad i \text{ no sum},
\]

(9)

where the subscript \( i \) of \( \hat{W} \) denotes the derivative with respect to \( \lambda_i \).

**Proof.** That (8) and (9) imply (7) is immediate by (6). On the other hand, applying the first derivative test to (7) yields (8).

Equations (8) are equilibrium equations for homogeneous deformations, which can also be derived from (3)_2. The pressure \( p \) can be eliminated in (8). The resulting two equations combined with (9) give stability conditions that do not depend explicitly on the load and can be used to determine, from the form of \( \hat{W} \) alone, whether a pure homogeneous deformation can be made stable in a pressure experiment.

By using the second derivative test, one can readily prove the next proposition concerning the local stability of pure homogeneous deformations.

**Proposition 4.** Under the \( C^1 \) norm, the pure homogeneous deformation (5) is locally stable under pressure \( p \) only if the matrix

\[
H = \left( \begin{array}{ccc}
\hat{W}_{11} & \hat{W}_{12} + p\lambda_3 & \hat{W}_{31} + p\lambda_2 \\
\hat{W}_{12} + p\lambda_3 & \hat{W}_{22} & \hat{W}_{23} + p\lambda_1 \\
\hat{W}_{31} + p\lambda_2 & \hat{W}_{23} + p\lambda_1 & \hat{W}_{33}
\end{array} \right)
\]

(10)

is positive semi-definite, and is locally stable if equations (8) are satisfied and the matrix \( H \) is positive definite, where the derivatives of \( \hat{W} \) are evaluated at \( (\lambda_1, \lambda_2, \lambda_3) \).

A class of pure homogeneous deformations of special importance to the present problem are symmetric deformations for which

\[
\lambda_1 = \lambda_2 = \lambda_3 \equiv \lambda.
\]

(11)
In this case, we have, by the symmetry of \( \hat{W} \),
\[
\hat{W}_1 = \hat{W}_2 = \hat{W}_3, \quad \hat{W}_{11} = \hat{W}_{22} = \hat{W}_{33}, \quad \hat{W}_{12} = \hat{W}_{23} = \hat{W}_{31}.
\]
The equilibrium equations (8) become
\[
\hat{W}_1(\lambda, \lambda, \lambda) + p\lambda^2 = 0. \tag{12}
\]
Also in this case, the positive semi-definiteness of \( H \) is equivalent to
\[
\begin{align*}
k_1 &\equiv \hat{W}_{11}(\lambda, \lambda, \lambda) + 2\hat{W}_{12}(\lambda, \lambda, \lambda) + 2p\lambda \geq 0, \\
k_2 &\equiv \hat{W}_{11}(\lambda, \lambda, \lambda) - \hat{W}_{12}(\lambda, \lambda, \lambda) - p\lambda \geq 0;
\end{align*} \tag{13}
\]
and the positive definiteness of \( H \) to the strict inequalities of (13). By Proposition 4, equation (12) and inequalities (13) are then necessary for the symmetric deformation (11) to be locally stable, and equation (12) and the strict inequalities (13) are sufficient for local stability.

As the symmetric deformations occupy a special position in this study owing to the symmetries of the load and material property, we cannot ignore the existence of asymmetric deformations. In fact, it is a focus of this work to examine the possibility of asymmetric solutions bifurcating from the symmetric solution branch. We now turn to this matter.

IV. BIFURCATION ANALYSIS

As the pressure \( p \) varies continuously in an interval, we assume that there exists a continuous family of symmetric homogeneous equilibrium deformations
\[
x_i = \lambda(p)X_i, \tag{14}
\]
where \( \lambda(p) \) satisfies the equilibrium equation (12). By (12) and (13), we also have
\[
\lambda'(p) = -\frac{\lambda^2}{k_1}.
\]
It then follows from Proposition 4 that the stretch \( \lambda \) must be decreasing in the pressure \( p \) at a stable symmetric deformation.

In studying the bifurcation of asymmetric solutions from the symmetric solution branch, we shall systematically exploit the symmetry property of the problem. We first note that the total energy function \( \phi \) defined in (6) is invariant under the permutation group \( S_3 \), that is,
\[
\phi(\gamma \lambda, p) = \phi(\lambda, p) \quad \forall \gamma \in S_3, \lambda \in \mathbb{R}_+^3, p \in \mathbb{R}.
\]
Here, we have written \((\lambda_1, \lambda_2, \lambda_3)\) as a column vector \( \lambda \) and identified \( S_3 \) with its linear representation:
\[
S_3 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\}.
\]

By a theorem\(^1\) of Schwartz [13], the function \( \phi \) can be expressed as a smooth function \( \psi \)

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1. Footnotes are found in the Notes section before the References.
of three invariant polynomials and $p$:

$$
\phi(\lambda_1, \lambda_2, \lambda_3, p) = \psi(u_1, u_2, u_3, p),
$$

where the polynomials

$$
u_1 \equiv \lambda_1 + \lambda_2 + \lambda_3, \quad u_2 \equiv \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1, \quad u_3 \equiv \lambda_1 \lambda_2 \lambda_3
$$

are called the generators of the invariant functions.

The same argument applies to $\hat{W}$, that is, there exists a smooth function $\hat{W}$, such that

$$
\hat{W}(\lambda_1, \lambda_2, \lambda_3) = \hat{W}(u_1, u_2, u_3).
$$

This symmetry is subsequently passed to the equilibrium equations (8), which can be rewritten as

$$
f(\lambda, p) = \mathbf{0},
$$

where $f: \mathbb{R}_+^3 \times \mathbb{R} \to \mathbb{R}^3$ is defined by

$$
f(\lambda, p) \equiv \begin{pmatrix}
\hat{W}_1 + p \lambda_2 \lambda_3 \\
\hat{W}_2 + p \lambda_3 \lambda_1 \\
\hat{W}_3 + p \lambda_1 \lambda_2
\end{pmatrix}.
$$

It is readily verified that $f$ is equivariant under $S_3$, that is,

$$
f(\gamma \lambda, p) = \gamma f(\lambda, p) \quad \forall \gamma \in S_3, \ \lambda \in \mathbb{R}_+^3, \ p \in \mathbb{R}.
$$

By a theorem of Poénaru [8], the function $f$ has the following representation

$$
f(\lambda_1, \lambda_2, \lambda_3, p) = f_1(u_1, u_2, u_3, p) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + f_2(u_1, u_2, u_3, p) \begin{pmatrix} \lambda_2 + \lambda_3 \\ \lambda_3 + \lambda_1 \\ \lambda_1 + \lambda_2 \end{pmatrix}
$$

$$
+ f_3(u_1, u_2, u_3, p) \begin{pmatrix} \lambda_2 \lambda_3 \\ \lambda_3 \lambda_1 \\ \lambda_1 \lambda_2 \end{pmatrix},
$$

where $f_i$ are smooth scalar functions, and the three column vectors on the right-hand side are equivariant polynomials that generate the equivariant functions. By (15), (16), and (18), these scalar functions are identified as

$$
f_1(u_1, u_2, u_3, p) = \tilde{W}_1(u_1, u_2, u_3), \quad f_2(u_1, u_2, u_3, p) = \tilde{W}_2(u_1, u_2, u_3), \quad f_3(u_1, u_2, u_3, p) = \tilde{W}_3(u_1, u_2, u_3) + p,
$$

where the subscript $i$ of $\tilde{W}$ denotes the derivative with respect to $u_i$.

By substituting (20) and (21) into (17), we find that the symmetric equilibrium deformation (14) satisfies

$$
\tilde{W}_1 + 2\lambda(p) \tilde{W}_2 + \lambda^2(p)(\tilde{W}_3 + p) = 0,
$$

where the derivatives of $\tilde{W}$ are evaluated at $(3\lambda(\ p), 3\lambda^2(\ p), \lambda^3(\ p))$. The above equation can also be derived from (12), (16), and (15).
Proposition 5. A symmetric solution $(\lambda_0, p_0) \equiv (\lambda_0, \lambda_0, \lambda_0, p_0)$ of (17) is a bifurcation point only if

$$k_1k_2 = 0,$$

where

$$k_1 \equiv 3(\tilde{W}_{11} + 4\lambda_0 \tilde{W}_{12} + 2\lambda_0^2 \tilde{W}_{13} + 4\lambda_0^2 \tilde{W}_{22} + 4\lambda_0^3 \tilde{W}_{23} + \lambda_0^4 \tilde{W}_{33})$$
$$\quad + 2(\tilde{W}_2 + \lambda_0 \tilde{W}_3 + p_0 \lambda_0),$$
$$k_2 \equiv -\tilde{W}_2 - \lambda_0 \tilde{W}_3 - p_0 \lambda_0. \quad (23)$$

Here and henceforth, the derivatives of $\tilde{W}$ are evaluated at $(3\lambda_0, 3\lambda_0^2, \lambda_0^3)$, unless stated otherwise.

Proof. By the implicit function theorem, an asymmetric solution branch can bifurcate from the symmetric solution branch at $(\lambda_0, p_0)$ only if the derivative $f_\lambda$ of $f$ with respect to $\lambda$ is not invertible at $(\lambda_0, p_0)$. A straightforward calculation, with the aid of (20), (21), and (15), shows that

$$f_\lambda(\lambda_0, p_0) = \frac{1}{3} \begin{pmatrix} k_1 + 2k_2 & k_1 - k_2 & k_1 - k_2 \\ k_1 - k_2 & k_1 + 2k_2 & k_1 - k_2 \\ k_1 - k_2 & k_1 - k_2 & k_1 + 2k_2 \end{pmatrix}.$$ \hfill \square

The conclusion follows from the observation that

$$\det f_\lambda(\lambda_0, p_0) = k_1k_2^2.$$

We note that the matrix $f_\lambda(\lambda_0, p_0)$ is identical to the matrix $H$ defined in (10) when the latter is evaluated at the symmetric solution $(\lambda_0, p_0)$. Also, the notation here has been so chosen that $k_1$ and $k_2$ defined in (23) agree with those defined in (13), the latter being again evaluated at $(\lambda_0, p_0)$. Propositions 4 and 5 then indicate that at a bifurcation point, the symmetric deformation may be on the verge of losing stability.

Condition (22) is in general not sufficient for $(\lambda_0, p_0)$ to be a bifurcation point. The derivation of sufficient conditions is often a more difficult task. In the sequel, we shall use singularity theory to derive such conditions for the two cases where the kernel of $f_\lambda(\lambda_0, p_0)$ has dimension one and two, respectively. For a comprehensive treatment of singularity theory, we refer the reader to Golubitsky and Schaeffer [2], and Golubitsky, Stewart and Schaeffer [4]. Here, we shall derive the normal forms of the bifurcation equation by using the techniques of singularity theory, and examine the stability of solution branches by using the stability conditions derived in the previous section.

Case I. $k_1 = 0, k_2 \neq 0$.

In this case, the kernel and co-kernel of $f_\lambda(\lambda_0, p_0)$ are both generated by

$$\mathbf{e} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \quad (24)$$

We first decompose the increment of the state variable $\lambda$ and the value of $f$ into orthogonal components with respect to $\mathbf{e}$, and write the equilibrium equation (17) as an equivalent system

$$Pf(\lambda_0 + x\mathbf{e} + h, p_0 + \mu) = 0, \quad h \cdot \mathbf{e} = 0, \quad \mathbf{e} \cdot f(\lambda_0 + x\mathbf{e} + h, p_0 + \mu) = 0, \quad (25)$$
where \( x \in \mathbb{R}, \ h \in \mathbb{R}^3, \ \mu \in \mathbb{R}, \) and \( P \) is the orthogonal projection of \( \mathbb{R}^3 \) onto the complement of \( e \). By the implicit function theorem and the condition \( k_2 \neq 0 \), we can formally solve equations (25) for \( h \), yielding a smooth function \( h(x, \mu) \). Further, by using the chain rule, we can show that

\[
h(0) = 0, \quad h_x(0) = 0.
\]  

(26)

Here and henceforth, we write the arguments of a function as a zero vector when it is evaluated at the origin. Although we cannot yet claim that \( h(x, \mu) = 0 \), we shall see below that this is indeed the case. Hence, this bifurcation leads to symmetric solutions that are \( S_3 \) invariant.

The original bifurcation problem (17) then reduces to

\[
g(x, \mu) = 0,
\]  

(27)

where

\[
g(x, \mu) \equiv e \cdot f(\lambda_0 + xe + h(x, \mu), p_0 + \mu).
\]  

(28)

Through the above procedure, known as Liapunov-Schmidt reduction, we now only need to solve a one-dimensional bifurcation problem. The new state variable \( x \) and bifurcation parameter \( \mu \) have been chosen so that the anticipated bifurcation point is translated to the origin.

Although it is usually impossible to find an explicit form for \( g \), the bifurcation diagram of (27) can be qualitatively determined by a normal form of \( g \), which is a polynomial function that preserves, in a certain sense, the solution set of (27) near the origin. This requires computing the derivatives of \( g \) at the origin. By applying the chain rule to (28) and using (26), we find that

\[
g(0) = 0, \quad g_x(0) = 0, \quad g_{\mu}(0) = 3\lambda_0^2, \quad g_{xx}(0) = 27k_3 + 54k_4 + 6k_5,
\]  

(29)

where

\[
k_3 \equiv \dot{W}_{111} = \dot{W}_{111} + 6\lambda_0 \dot{W}_{112} + 3\lambda_0^2 \dot{W}_{113} + 12\lambda_0^2 \dot{W}_{122} + 12\lambda_0^3 \dot{W}_{123} + 3\lambda_0^4 \dot{W}_{133} + 8\lambda_0^3 \dot{W}_{222} + 12\lambda_0^4 \dot{W}_{223} + 6\lambda_0^5 \dot{W}_{233} + \lambda_0^6 \dot{W}_{333},
\]  

(30)

\[
k_4 \equiv \frac{1}{2} (-\dot{W}_{111} + \dot{W}_{112}) = \dot{W}_{12} + \lambda_0 \dot{W}_{13} + 2\lambda_0 \dot{W}_{22} + 3\lambda_0^2 \dot{W}_{23} + \lambda_0^3 \dot{W}_{33},
\]  

\[
k_5 \equiv \frac{1}{2} (\dot{W}_{111} - 3\dot{W}_{112}) + \dot{W}_{123} + p_0 = \dot{W}_3 + p_0.
\]  

**Proposition 6.** If

\[
g(0) = 0, \quad g_x(0) = 0, \quad g_{\mu}(0) \neq 0, \quad g_{xx}(0) \neq 0,
\]  

(31)

then equation (27), and therefore (17), is equivalent to

\[
\mu + \epsilon x^2 = 0,
\]  

(32)

where

\[
\epsilon \equiv \text{sgn}(g_{\mu}(0)g_{xx}(0)).
\]
even if (31) does not hold, since the essential conditions for a single solution branch near 
$(\lambda_0, p_0)$ are (31)$_{1-3}$, which by (29) always hold.

**Case II.** $k_1 \neq 0, k_2 = 0$.

In this case, the kernel and co-kernel of $f_4(\lambda_0, p_0)$ are of dimension two. The Liapunov-Schmidt scheme will reduce the dimension of the bifurcation problem by only one. Nevertheless, the reduced two-dimensional problem possesses the symmetry property that can be described by the action of $S_3$ group on the complex plane. This enables us to use the known results for such a problem to find bifurcation conditions.

The kernel and co-kernel of $f_4(\lambda_0, p_0)$ can be spanned by two orthonormal vectors

$$e_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \quad e_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}. \quad (33)$$

Here we have chosen $e_1$ and $e_2$ in such a way that the reduced bifurcation equation is equivariant under the action of $S_3$ on $\mathbb{C}$. The vector $e$ defined in (24) is now orthogonal to the kernel and the co-kernel.

We first examine the projection of equation (17) onto the complement of the co-kernel:

$$e \cdot f(\lambda_0 + xe_1 + ye_2 + he, p_0 + \mu) = 0, \quad (34)$$

where $x \in \mathbb{R}, y \in \mathbb{R}, h \in \mathbb{R}, \mu \in \mathbb{R}$. By the implicit function theorem and the condition $k_1 \neq 0$, one can solve (34) for $h$, which yields a smooth function $h(x, y, \mu)$. We can identify $(x, y)$ with a complex number $z$ through $z = x + yi$, $i$ being the imaginary unit, and regard $h$ as a function from $\mathbb{C} \times \mathbb{R}$ to $\mathbb{R}$. It can be readily verified that $h$ is invariant under the action of $S_3$ on $\mathbb{C}$, that is,

$$h(\gamma z, \mu) = h(z, \mu) \quad \forall \gamma \in S_3, \ z \in \mathbb{C}, \ \mu \in \mathbb{R}. \quad (35)$$

Here we have identified $S_3$ with its action on $\mathbb{C}$, for example,

$$\gamma_{(12)} z = \bar{z}, \ \gamma_{(123)} z = e^{\frac{2\pi i}{3}} z, \quad (36)$$

where $\gamma_{(12)}$ is the action of the interchange permutation of the first two elements, $\gamma_{(123)}$ the action of cyclic permutation, and $\bar{z}$ the complex conjugate of $z$. The derivatives of $h$ at the origin can be calculated by applying the chain rule to (34), yielding

$$h(0) = 0, \quad h_x(0) = h_y(0) = 0, \quad h_\mu(0) = -\frac{\lambda_0^2}{k_1}, \quad (37)$$

$$h_{xx}(0) = h_{yy}(0) = \frac{3k_4 + k_5}{3k_1}, \quad h_{xy}(0) = 0.$$ 

To complete the Liapunov-Schmidt reduction, we substitute $h(z, \mu)$ into the projection of (17) onto the co-kernel, which is also identified with $\mathbb{C}$ through the base vectors $e_1$ and $e_2$. This leads to a smooth function $g : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ defined by

$$g(x + yi, \mu) \equiv (e_1 + e_2 i) \cdot f(\lambda_0 + xe_1 + ye_2 + h(x + yi, \mu)e, p_0 + \mu). \quad (38)$$
By (36), (38), (35), (33), and (19), we find that
\[ g(y_{(12)}z, \mu) = g(\bar{z}, \mu) \]
\[ = (e_1 + e_2 i) \cdot f(\lambda_0 + xe_1 - ye_2 + h(z, \mu)e, p_0 + \mu) \]
\[ = (e_1 - e_2 i) \cdot f(\lambda_0 + xe_1 + ye_2 + h(z, \mu)e, p_0 + \mu) \]
\[ = g(z, \mu) \]
\[ = y_{(12)} g(z, \mu). \]

The same relation holds for other members of \( S_3 \), in other words, \( g \) is equivariant under the action of \( S_3 \) on \( \mathbb{C} \):
\[ g(yz, \mu) = y g(z, \mu) \quad \forall \ y \in S_3, \ z \in \mathbb{C}, \ \mu \in \mathbb{R}. \tag{39} \]

Now the problem becomes one of studying the solutions of
\[ g(z, \mu) = 0, \tag{40} \]
where \( g \) satisfies (39). This problem has been studied by Golubitsky and Schaeffer [3]. In the next two propositions we state some of their results without giving proofs.

**Proposition 7.** There exist smooth real valued functions \( a \) and \( b \), such that
\[ g(z, \mu) = a(|z|^2, \text{Re } z^3, \mu) z + b(|z|^2, \text{Re } z^3, \mu) \bar{z}^2. \tag{41} \]

We note that in the above representation, \(|z|^2\) and \( \text{Re } z^3 \) are two polynomial generators of the invariants, and \( z \) and \( \bar{z}^2 \) are two polynomial generators of the equivariants.

To find normal forms of (41), we need to compute the coefficients \( a \) and \( b \) and their derivatives at the origin. By (38) and (41), we have
\[ e_1 \cdot f(\lambda_0 + xe_1 + ye_2 + h(x + yi, \mu)e, p_0 + \mu) = xa(x^2 + y^2, x^3 - 3xy^2, \mu) \]
\[ + (x^2 - y^2)b(x^2 + y^2, x^3 - 3xy^2, \mu), \tag{42} \]
\[ e_2 \cdot f(\lambda_0 + xe_1 + ye_2 + h(x + yi, \mu)e, p_0 + \mu) = ya(x^2 + y^2, x^3 - 3xy^2, \mu) \]
\[ - 2xyb(x^2 + y^2, x^3 - 3xy^2, \mu). \]

By repeatedly applying the chain rule to (42) and using (37), we find that
\[ a(0) = 0, \quad b(0) = -\frac{k_5}{\sqrt{6}}, \quad a_1(0) = \frac{k_6}{2} - \frac{(3k_4 + k_5)^2}{6k_1}, \quad a_2(0) = \frac{k_6^2(3k_4 + k_5)}{k_1} - \lambda_0, \tag{43} \]
where the subscript \( i \) of \( a \) denote the derivative with respect to its \( i \)th argument, parameters \( k_i, i \leq 5 \) are defined in (23) and (30), and \( k_6 \) given by
\[ k_6 = \frac{1}{6} \tilde{W}_{1111} - \frac{2}{3} \tilde{W}_{1112} + \frac{1}{2} \tilde{W}_{1122} = \tilde{W}_{22} + 2\lambda_0 \tilde{W}_{23} + \lambda_0^2 \tilde{W}_{33}. \]

**Proposition 8.** If
\[ a(0) = 0, \quad b(0) \neq 0, \quad a_3(0) \neq 0, \tag{44} \]
then equation (40) is equivalent to

\[ \mu z + \epsilon z^2 = 0, \]  

(45)

where

\[ \epsilon \equiv \text{sgn}(b(0)a_3(0)). \]

Equation (45) has a trivial solution \( z = 0 \) and three nontrivial solutions

\[ z = -\epsilon \mu, \quad z = -\epsilon e^{\frac{2\pi i}{3}} \mu, \quad z = -\epsilon e^{-\frac{2\pi i}{3}} \mu. \]

The trivial solution corresponds to the symmetric deformations, and the nontrivial solutions to asymmetric deformations with two equal principal stretches. The three nontrivial solutions are all group related, that is, they can be generated from just one by the action of \( S_3 \) group.

The above results can be also obtained by a direct calculation. Let \( z = \xi e^{\theta i}, \xi \in \mathbb{R}, \theta \in \mathbb{R}, 0 \leq \theta < \pi \). By (41), equation (40) can be rewritten as

\[ \xi [a(\xi^2, \xi^3 \cos 3\theta, \mu) + \xi b(\xi^2, \xi^3 \cos 3\theta, \mu) e^{-3\theta i}] = 0. \]

(46)

Because \( b(0) \neq 0 \), a solution of (46) near the origin must satisfy either \( \xi = 0 \) or \( 3\theta = n\pi, n \) being an integer. The former condition gives the symmetric solution, and the latter, written as \( \theta = 0, \pi/3, 2\pi/3 \), leads to the three asymmetric solutions. Indeed, for a fixed \( n = 0, 1, 2 \), the implicit function theorem and condition \( b(0) \neq 0 \) ensure that the equation

\[ a(\xi^2, \xi^3 \cos n\pi, \mu) + \xi b(\xi^2, \xi^3 \cos n\pi, \mu) \cos n\pi = 0 \]

(47)

has a unique solution \( \xi = \xi(\mu) \) near the origin, with

\[ \xi'(0) = -\frac{a_3(0)}{b(0)} \cos n\pi. \]

(48)

The corresponding asymmetric solution of (40) is then

\[ z = \xi(\mu)e^{\frac{n\pi i}{3}}. \]

(49)

By Proposition 4 and (13), all deformations near \( \lambda_0 \) are unstable if \( k_1 < 0 \). The next proposition gives the stability properties of the solutions when \( k_1 > 0 \). We note that the same conclusions have been reached in [3] by examining the linear orbital stabilities of the solutions of (45). The following alternative proof checks directly the stabilities of the solutions of (49).

**Proposition 9.** Suppose that \( k_1 > 0 \) and conditions (44) hold. Then all equilibrium deformations pertinent to Proposition 8 are unstable near \( \lambda_0 \), except for those symmetric solutions \( \lambda = (\lambda, \lambda, \lambda) \) that satisfy

\[ (\lambda - \lambda_0)a_3(0) \leq 0. \]

(50)

**Proof.** First consider an asymmetric solution (49). The corresponding asymmetric deformation is

\[ \lambda(\mu) = \lambda_0 + \xi(\mu) \left( e_1 \cos \frac{n\pi}{3} + e_2 \sin \frac{n\pi}{3} \right) + h \left( \xi(\mu) \cos \frac{n\pi}{3}, \xi(\mu) \sin \frac{n\pi}{3}, \mu \right) e. \]

(51)

We wish to show that the matrix \( H \) defined in (10) is not positive semi-definite at \( \lambda = \lambda(\mu) \), \( p = p_0 + \mu \) for small nonzero \( \mu \). Let

\[ v = e_1 \cos \frac{n\pi}{3} + e_2 \sin \frac{n\pi}{3}. \]
By using (51), (48), and (37), we find that
\[
v \cdot H v = v \cdot f_3(\lambda(\mu), p_0 + \mu)v
= \mu v \cdot [f_{\lambda\lambda}(\lambda_0, p_0)\lambda'(0) + f_{\lambda p}(\lambda_0, p_0)]v + o(\mu)
= -\mu a_3(0) + o(\mu).
\]

Further, let
\[
v = e_1 \sin \frac{n\pi}{3} - e_2 \cos \frac{n\pi}{3}.
\]

A similar calculation shows that
\[
v \cdot H v = 3\mu a_3(0) + o(\mu).
\]

Hence, the quadratic form of \( H \) can be made negative on either side of the bifurcation point along the asymmetric solution. To complete the proof, it remains to show the stability property of the symmetric solution \( \lambda = (\lambda, \lambda, \lambda) \). By Proposition 4 and the assumption that \( k_1 > 0 \) at \( \lambda = \lambda_0 \), we only need to check the sign of \( k_2 \) near \( \lambda = \lambda_0 \). In light of (13)\(_2\) and (12), we define
\[
k_2(\lambda) \equiv \hat{W}_{11}(\lambda, \lambda, \lambda) - \hat{W}_{12}(\lambda, \lambda, \lambda) + \frac{\hat{W}_1(\lambda, \lambda, \lambda)}{\lambda}.
\]

By (30), (13)\(_1\), (12), and (43), we find that
\[
k_2'(|\lambda_0|) = -\frac{k_1 a_3(0)}{\lambda_0^2}.
\]

Then for \( \lambda \) near \( \lambda_0 \), we have
\[
k_2(\lambda) = (\lambda - \lambda_0)k_2'(\lambda_0) + o(\lambda - \lambda_0) = -\frac{(\lambda - \lambda_0)k_1 a_3(0)}{\lambda_0^2} + o(\lambda - \lambda_0),
\]
which shows that \( k_2(\lambda) \) is nonnegative if and only if (50) holds.

Although all the asymmetric solutions are unstable locally in the above case, we should not ignore these solutions. They are analogous to those of the Rivlin [11] cube problem, in which the symmetric solution branch becomes unstable at a bifurcation point where the symmetric solution branch intersects three asymmetric solution branches. All three asymmetric solution branches are unstable near the bifurcation point, but become stable when the equal stretches exceed a certain value, as sketched in Figure 1. Hence, the cube will undergo a jump from the symmetric deformation to a stable asymmetric deformation, as indicated by the arrow in Figure 1. Similarly, in the present case, if the growth conditions in Proposition 1 are satisfied, the body must undergo a jump from the symmetric deformation to a stable asymmetric deformation as \( p \) passes \( p_0 \).

We now state a proposition for another normal form that corresponds to a degenerate bifurcation. The proof can be found again in [3].

**Proposition 10.** If
\[
a(0) = 0, \quad b(0) = 0, \quad a_1(0) \neq 0, \quad a_3(0) \neq 0,
\]
\[
a_1(0)b_2(0) - a_2(0)b_1(0) \neq 0, \quad a_1(0)b_3(0) - a_3(0)b_1(0) \neq 0,
\]
(52)
then equation (40) is equivalent to
\[ (|z|^2 + \epsilon_1 \mu)z + (\epsilon_2 |z|^2 + m \Re z^3)\zbar^2 = 0, \] (53)
where
\[ \epsilon_1 \equiv \text{sgn}(a_1(0)a_3(0)), \quad \epsilon_2 \equiv \epsilon_1 \text{sgn}(a_3(0)b_1(0) - a_1(0)b_3(0)), \]
\[ m \equiv \frac{a_3^2(0)[a_1(0)b_2(0) - a_2(0)b_1(0)]}{[a_1(0)b_3(0) - a_3(0)b_1(0)]^2}. \]

It is readily verified that near the origin equation (53) has a trivial solution $z = 0$, which corresponds to the symmetric deformations, and, when $\epsilon_1 \mu < 0$, six nontrivial solutions, which correspond to asymmetric deformations. These asymmetric deformations again have partial symmetry with two equal stretches and are group related.

The bifurcation diagrams of (53) are illustrated in [3]. Also included in [3] is a discussion of the universal unfolding of (53). Because there is only one bifurcation parameter $p$ in our problem, the bifurcation conditions $k_2 = 0$ and $b(0) = 0$ make the bifurcation (53) unlikely to occur for a real material. Only an ideal material for which a certain equation is satisfied exactly can have this bifurcation. However, if the equation is satisfied approximately for a real material, the corresponding bifurcation can be found by introducing certain perturbation parameters into (53). A universal unfolding gives all possible perturbed bifurcations, and hence covers the entire family of solutions in a neighborhood of the one for the ideal case. The perturbations may be incurred by imperfections of the constitutive equation, of the boundary conditions, and for other problems, of the geometry. A particularly interesting feature associated with the
universal unfolding of (53) is the existence of a secondary bifurcation to totally asymmetric solutions with three unequal principal stretches.

The linear orbital stabilities of the solution branches of (53) are given in [3] for one combination of the parameters. In the next proposition, we give the stability properties of the solutions by directly checking the stability conditions for the solutions of (40).

**Proposition 11.** Suppose that \(k_1 > 0\) and conditions (52) hold. The matrix \(H\) defined in (10) is positive semi-definite at a symmetric equilibrium deformation \(\lambda = (\lambda, \lambda, \lambda)\) near \(\lambda_0\) if and only if (50) holds, and is positive semi-definite at an asymmetric deformation near \(\lambda_0\) if and only if \(k_6 \geq 0\).

**Proof.** We only need to analyze asymmetric deformations since the proof for symmetric deformations is identical to that of Proposition 9. An asymmetric solution of (40) can again be expressed as

\[
z = \xi e^{zi} ,
\]

where \(n\) is an integer, and \(\xi \in \mathbb{R}\) satisfies (47). Because \(a_3(0) \neq 0\), equation (47) has a unique solution \(\mu = \mu(\xi)\) near the origin, with

\[
\mu'(0) = 0, \quad \mu''(0) = -\frac{2a_1(0)}{a_3(0)} .
\]  

(54)

The corresponding asymmetric equilibrium deformation is

\[
\lambda(\xi) = \lambda_0 + \xi \left( e_1 \cos \frac{n\pi}{3} + e_2 \sin \frac{n\pi}{3} \right) + h \left( \xi \cos \frac{n\pi}{3}, \xi \sin \frac{n\pi}{3}, \mu(\xi) \right) .
\]

(55)

For \(v \in \mathbb{R}^3\), we then have

\[
v \cdot Hv = v \cdot f_{\lambda}(\lambda(\xi), p_0 + \mu(\xi))v
\]

\[
= v \cdot f_{\lambda}(\lambda_0, p_0)v + O(\xi)
\]

\[
= \frac{k_1}{3} |e \cdot v|^2 + O(\xi).
\]

For sufficiently small \(\xi\), the last expression is positive unless \(e \cdot v = 0\). When \(e \cdot v = 0\), we expand the quadratic form of \(H\) to the second order of \(\xi\) and make use of (55), (37), (54), (52), and (43) to arrive at

\[
v \cdot Hv = \frac{1}{2} \xi^2 v \cdot [f_{\lambda,\lambda}(\lambda_0, p_0)[\lambda'(0) \otimes \lambda'(0)]
\]

\[
+ f_{\lambda,\lambda}(\lambda_0, p_0)[\lambda''(0)] + f_{\lambda,\mu}(\lambda_0, p_0)\mu''(0)]v + o(\xi^2)
\]

\[
= \xi^2 k_6 \left[ v \cdot \left( e_1 \cos \frac{n\pi}{3} + e_2 \sin \frac{n\pi}{3} \right) \right]^2 + o(\xi^2).
\]

The conclusion then follows.

\(\square\)

As a concluding remark, we comment on the possible deformation path as \(p\) passes \(p_0\) in this last case. By (50) and (37), the symmetric solution branch suffers a change in stability at \(p = p_0\), that is, it is stable at one side and unstable at the other side of \(p = p_0\). Further, if \(a_1 < 0\), it follows from (54), (50), and (37) that the asymmetric solution branches exist only
on the side where the symmetric solution branch is stable. Hence, as $p$ passes $p_0$, either a
jump takes place so that the deformation continues on a stable solution branch a finite distance
from $\lambda_0$, or a stable deformation no longer exists. Finally, if $a_1 > 0$, the asymmetric solutions
and stable symmetric solutions are on the different sides of the bifurcation point. Also, by
(43)_3, the parameter $k_6$ must be positive. It is then possible that the asymmetric solutions are
stable and the symmetric deformation bifurcates continuously to asymmetric deformations as
$p$ passes $p_0$.

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NOTE

1. The theorem in [13] is for $C^\infty$ functions. This may require additional smoothness of $\phi$ to that previously
assumed. The same argument applies to Poénaru's [8] theorem.

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